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Intrinsic formulation for elastic line deformed on a surface by external field in the Minkowski 3-space $\mathbb{E}_1^3$

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Abstract

In this work, we developed intrinsic formulation for elastic line deformed on a surface by an external field in the Minkowski 3-space.

Keywords: Elastic lines; Variational problem; Minkowski 3-space

1. Introduction

In this section, we will give some fundamental definitions and theorems.

Definition 1.1. $\mathbb{E}^n$ with the metric

$$\langle v, w\rangle = -\sum_{i=1}^{\nu} v_i w_i + \sum_{j=\nu+1}^{n} v_j w_j, \quad v, w \in \mathbb{E}^n,\ 0 \leqslant \nu \leqslant n,$$

is called semi-Euclidean space and is denoted by $\mathbb{E}_\nu^n$, where is called the index of the metric. For $n = 3$, $\mathbb{E}_1^3$ is called Minkowski 3-space [2].

Definition 1.2. Let $\mathbb{E}_\nu^n$ be a semi-Euclidean space furnished with a metric tensor $\langle\, ,\rangle$. A vector v to $\mathbb{E}_\nu^n$ is called

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spacelike if $\langle v, v\rangle > 0$ or $v = 0$,
null if $\langle v, v\rangle = 0$ and $v \neq 0$,
timelike if $\langle v, v\rangle < 0$ [2].

Definition 1.3. Apart from the Frenet frame $\{T, n, b\}$, there also exists a second frame at every point of curve α. At a point $\alpha(s)$ of α, let T denote the unit tangent vector to α, N the unit normal to M, and

$$N \times T = \varepsilon Q(s), \quad \varepsilon = \pm 1, \tag{1}$$

respectively, then with the respect to which inner product $\{T, Q, N\}$ gives an orthonormal basis in $\mathbb{E}_1^3$. If M is spacelike surface, $T \times Q = N$, $Q \times N = -T$, $N \times T = -Q$. Similarly if M is timelike surface, $T \times Q = N$, $Q \times N = \pm T$, $N \times T = -Q$ [3].

Theorem 1.1. *Let M be a surface in $\mathbb{E}_1^3$ and α a curve on M. The analogue of the Frenet–Serret formulas is given by*

$$\begin{bmatrix} T' \\ Q' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 k_g & \varepsilon_3 k_n \\ -\varepsilon_1 k_g & 0 & \varepsilon_3 \tau_g \\ -\varepsilon_1 k_n & -\varepsilon_2 \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ Q \\ N \end{bmatrix},$$

where $\langle T, T\rangle = \varepsilon_1$, $\langle Q, Q\rangle = \varepsilon_2$, $\langle N, N\rangle = \varepsilon_3$. Here $k_g(s) = \langle T'(s), Q(s)\rangle$, $\tau_g(s) = \langle Q'(s), N(s)\rangle$ and $k_n(s) = \langle II(T(s), T(s)), N\rangle$ are respectively the geodesic curvature, the geodesic torsion and the normal curvature [4].

Remark 1.1. Let $x(u, v)$ be the timelike surface, having parameter curves which are perpendicular to each other passing through point $\alpha(s)$ of any curve α. Then, the geodesic curvature is

$$k_g = (k_g)_1 \cosh\gamma - (k_g)_2 \sinh\gamma - \frac{d\gamma}{ds}.$$

Here,

$$(k_g)_1 = -\frac{1}{2}\frac{E_v}{|E||G|^{1/2}}, \qquad (k_g)_2 = \frac{1}{2}\frac{G_u}{|E|^{1/2}|G|}.$$

The normal curvature is $k_n = k_1 \cosh^2\gamma - k_2 \sinh^2\gamma$ and the geodesic torsion is $\tau_g = (k_2 - k_1) \times \cosh\gamma \sinh\gamma$ [3].

Definition 1.4. Let h denote the second fundamental form of M in the Minkowski 3-space. With respect to a Lorentzian frame field (e_1, e_2, e_3), h is represented by the matrix h_{ij}, where

$$h_{ij} = -\langle D_{e_i} e_j, e_3\rangle, \tag{2}$$

D denoting covariant differentiation in L^3 [4].

Definition 1.5. Let α, be a curve on a surface M in $\mathbb{E}_1^3$, parametrized by arc length s, $0 \leqslant s \leqslant l$. An elastic line of invariant length l, may be defined by its stress energy I_1,

$$I_1 = \frac{1}{2} b \int_0^l \kappa^2 \, ds,$$

where b is the Hooke's law bending constant and $\kappa^2(s)$ the square curvature at arc length s along the line [1,5].

2. Intrinsic method

If elastic line is exposed to a static force field, it has a trajectory that minimizes the sum of its elastic energy and its energy of interaction with the field. The equilibrium trajectory are the extrema of the sum of stress and potential energies. The path of the elastic line must satisfy a differential equation on the nonnull semi-Euclidean surface, which is derived by variational methods in the Minkowski 3-space $\mathbb{E}_1^3$. The problem is to minimize the energy K,

$$K = \int_0^l \left(\frac{1}{2}b\kappa^2 - \gamma w\right) ds,$$

among elastic lines with trajectories $x[u(s), v(s)]$ of fixed length l and arc length s, $0 \leqslant s \leqslant l$, contained in nonnull semi-Euclidean surface $x(u, v)$ in the Minkowski 3-space $\mathbb{E}_1^3$. The curvature at distance s along the trajectory is $\kappa(u(s), v(s))$. The first term of K represents elastic bending energy. Additionally, a segment ds of the elastic line at arc length s along the trajectory experiences a force tangential to the nonnull surface characterized by the nonnull surface gradient potential energy $-\gamma w\, ds$, where γ is a constant measuring the strength of the external field and $\omega(u, v)$ gives its shape in the Minkowski 3-space $\mathbb{E}_1^3$.

Now, assume α is the trajectory in the nonnull semi-Euclidean surface that minimizes K and α lies in a coordinate patch $(u, v) \to x(u, v)$ of nonnull semi-Euclidean surface M. Then, α is expressed as $\mathbb{E}_1^3$

$$\alpha(s) = x\big(u(s), v(s)\big), \quad 0 \leqslant s \leqslant l,$$

with

$$T(s) = \alpha'(s). \tag{3}$$

The vector N is the unit nonnull semi-Euclidean surface, normal

$$N(s) = \frac{x_u(s) \times x_v(s)}{|EG - F^2|^{1/2}} \tag{4}$$

and x_u and x_v are nonnull surface tangent vectors along the u-parameter and v-parameter curves. Thus T and Q are denoted

$$T = u'x_u + v'x_v, \qquad Q(s) = p(s)x_u + q(s)x_v,$$

where suitable scalar functions $p(s)$ and $q(s)$. p and q are obtained with aid of (4) and (1). For spacelike surface,

$$p = \frac{u'F + v'G}{|EG - F^2|^{1/2}}, \qquad q = -\frac{u'E + v'F}{|EG - F^2|^{1/2}}. \tag{5}$$

On timelike surface for timelike arc α, the functions p and q is the same with spacelike surface.

Next, we must define variational fields for our problem. In order to obtain variational timelike or spacelike arcs of length l, it is generally necessary to extend α to an arc α^* defined for $0 \leqslant s \leqslant l^*$, with $l^* > l$, but sufficiently close to l so that α^* lies in the coordinate patch. Let $\mu(s)$, $0 \leqslant s \leqslant l^*$, be a scalar function. Define

$$\eta(s)=\mu(s)p(s), \qquad \xi(s)=\mu(s)q(s), \qquad \eta(s)x_u+\xi(s)x_v=\mu(s)Q(s). \tag{6}$$

Assume also that

$$\mu(0)=0, \qquad \mu'(0)=0. \tag{7}$$

Now define

$$\beta(\sigma;t)=x\big(u(\sigma)+t\eta(\sigma), v(\sigma)+t\xi(\sigma)\big) \tag{8}$$

for $0\leqslant\sigma\leqslant l^*$. For $|t|<\varepsilon$ (where $\varepsilon>0$ depends upon the choice of α^* and of μ), the point $\beta(\sigma;t)$ lies in the coordinate patch. For fixed t, $\beta(\sigma;t)$ gives an arc with the same initial point and initial direction as α. For $t=0$, $\beta(\sigma;0)$ is the same as α^* and σ is arc length. For $t\neq 0$, the parameter σ is not arc length in general.

For fixed t, $|t|<\varepsilon$, let $L^*(t)$ denote the length of the arc $\beta(\sigma;t)$, $0\leqslant\sigma\leqslant l^*$. Then

$$L^*(t)=\int_0^{l^*}\sqrt{\left|\left\langle\frac{\partial\beta}{\partial\sigma}(\sigma;t), \frac{\partial\beta}{\partial\sigma}(\sigma;t)\right\rangle\right|}\,d\sigma \tag{9}$$

with

$$L^*(0)=l^*>l. \tag{10}$$

It is clear from (8) and (9) that $L^*(t)$ is continuous. In particular, it follows from (10) that

$$L^*(t)>\frac{l+l^*}{2}>l, \quad |t|<\varepsilon_1, \tag{11}$$

for a suitable ε^* satisfying $0<\varepsilon^*\leqslant\varepsilon$. Because of (11), we can restrict $\beta(\sigma;t)$, $0\leqslant|t|<\varepsilon^*$ to an arc of length l by restricting the parameter σ to an interval $0\leqslant\sigma\leqslant\lambda(t)\leqslant l^*$, by requiring

$$\int_0^{\lambda(t)}\sqrt{\left|\left\langle\frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle\right|}\,d\sigma=l. \tag{12}$$

Note that $\lambda(0)=l$,

$$\left.\frac{d\lambda}{dt}\right|_{t=0}=\varepsilon_1\int_0^l \mu k_g\,ds. \tag{13}$$

The proof of (13) and of other results below will depend on calculations from (8) such as

$$\left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}=T, \quad 0\leqslant\sigma\leqslant l, \tag{14}$$

which gives

$$\left.\frac{\partial^2\beta}{\partial\sigma^2}\right|_{t=0}=T'=\varepsilon_2 k_g Q+\varepsilon_3 k_n N. \tag{15}$$

Also, it follows from (6) that

$$\left.\frac{\partial\beta}{\partial t}\right|_{t=0}=\mu Q. \tag{16}$$

Using (6), the second differentiation of (16) gives

$$\frac{\partial^2\beta}{\partial t\partial\sigma} = -\varepsilon_1\mu k_g T + \mu' Q + \varepsilon_3\mu\tau_g N. \tag{17}$$

To prove (13), differentiate (12) with respect to t, remembering that l is constant, and evaluate at $t=0$ using (14) and (17), with $\lambda(0)=l$. We get

$$\lambda'(t)\sqrt{\left|\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle\right|}\,\Bigg|_{s=\lambda(t)} + \int_0^{\lambda(t)} \frac{\partial}{\partial t}\left(\sqrt{\left|\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle\right|}\right) ds = 0.$$

At $t=0$,

$$\lambda'(0) = -\int_0^l \frac{\partial}{\partial t}\left(\sqrt{\left|\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle\right|}\right) ds = \varepsilon_1\mu k_g.$$

The energy of an elastic line with trajectory $\beta(\sigma;t)$ is given by

$$K(t) = \frac{1}{2}bI_1(t) - \gamma I_2(t), \tag{18}$$

$$I_1(t) = \int_0^l \kappa^2\,ds, \tag{19}$$

$$I_2(t) = \int_0^l w\,ds. \tag{20}$$

Due to $\alpha(s)$ is the minimum energy trajectory, $K'(t)$ is vanish at $t=0$. So, differentiating the integrals I_1 and I_2 require. In calculating. Here, $I_1(t)$ denote the total square curvature of the arc $\beta(\sigma;t)$, $0\leqslant\sigma\leqslant\lambda(t)$, $|t|<\varepsilon^*$. Then,

$$I_1'(t) = \lambda'(t)\left[\kappa^2\left(\sqrt{\left|\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle\right|}\right)\right]\Bigg|_{s=\lambda(t)} + \int_0^{\lambda(t)}\left(\frac{\partial}{\partial t}\right)\left[\kappa^2\left(\sqrt{\left|\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle\right|}\right)\right] ds.$$

At $t=0$,

$$I_1'(0) = \lambda'(0)\kappa_l^2 + \int_0^l\left(\frac{\partial\kappa^2}{\partial t}\bigg|_{t=0}\right)ds + \int_0^l \kappa^2\left(\sqrt{\left|\left\langle\frac{\partial^2\beta}{\partial t\partial\sigma}\bigg|_{t=0},\frac{\partial^2\beta}{\partial t\partial\sigma}\bigg|_{t=0}\right\rangle\right|}\right) ds.$$

After complicated computations,

$$\begin{aligned} I_1'(0) = {}& \varepsilon_1\int_0^l \mu k_g\,ds\left\{\left|\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2\right|\right\}_{\sigma=\lambda(0)} \\ & + 2\int_0^l k_g\left(\mu'' - \varepsilon_1\varepsilon_2\mu k_g^2 - \varepsilon_2\varepsilon_3\mu\tau_g^2\right)\frac{|\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2|}{\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2}\,ds \end{aligned}$$

$$+ 2\int_0^l k_n(2\varepsilon_3\mu'\tau_g - \varepsilon_1\varepsilon_3\mu k_g k_n + \varepsilon_3\mu\tau_g')\frac{|\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2|}{\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2}\,ds$$
$$+ 3\varepsilon_1\int_0^l \mu k_g\left|\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2\right|ds. \tag{21}$$

With differentiating of (20) at $t = 0$,

$$I_2'(0) = \int_0^l \left[\left(\frac{\partial w}{\partial t}\right)\bigg|_{t=0} - \varepsilon_1\mu k_g\big(w - w(l)\big)\right]ds. \tag{22}$$

As a function of coordinates along β

$$w = w\big[u(\sigma) + t\mu(\sigma)p(\sigma), v(\sigma) + t\mu(\sigma)q(\sigma)\big], \tag{23}$$

so that

$$\left(\frac{\partial w}{\partial t}\right)\bigg|_{t=0} = \mu\left[p\left(\frac{\partial w}{\partial u}\right) + q\left(\frac{\partial w}{\partial v}\right)\right]. \tag{24}$$

Eventually,

$$I_2'(0) = \int_0^l \mu\left[p\left(\frac{\partial w}{\partial u}\right) + q\left(\frac{\partial w}{\partial v}\right) - \varepsilon_1 k_g\big(w - w(l)\big)\right]ds. \tag{25}$$

2.1. *Intrinsic equations for a elastic line deformed on a timelike surface for timelike arc α*

From (21),

$$I_1'(0) = \int_0^l \mu\big\{2k_g'' - 2k_n\tau_g' - 4k_n'\tau_g + k_g\big(-k_g^2(l) - k_n^2(l) - k_g^2 - k_n^2 - 2\tau_g^2\big)\big\}\,ds$$
$$+ 2\mu'(l)k_g(l) - 2\mu(l)k_g'(l) + 4\mu(l)k_n(l)\tau_g(l) \tag{26}$$

and from (25)

$$I_2'(0) = \int_0^l \mu\left[p\left(\frac{\partial w}{\partial u}\right) + q\left(\frac{\partial w}{\partial v}\right) + k_g\big(w - w(l)\big)\right]ds. \tag{27}$$

From (18), (26), (27), for all choices of the function $\mu(s)$ satisfying (7), with arbitrary values of $\mu(l)$ and $\mu'(l)$, and $K'(0) = 0$, the path of timelike arc $\alpha(s)$ must satisfy two boundary conditions and differential equation

(1) $k_g(l) = 0$,

(2) $k_g'(l) = 2k_n(l)\tau_g(l)$,

(3) $$\frac{1}{2}b\big[2k_g'' - 2k_n\tau_g' - 4k_n'\tau_g + k_g\big(-k_g^2(l) - k_n^2(l) - k_g^2 - k_n^2 - 2\tau_g^2\big)\big]$$
$$- \gamma\left[p\left(\frac{\partial w}{\partial u}\right) + q\left(\frac{\partial w}{\partial v}\right) + k_g\big(w - w(l)\big)\right] = 0. \tag{28}$$

2.2. Intrinsic equations for a elastic line deformed on an timelike surface for spacelike arc α

(i) In case $k_g^2 < k_n^2$,

$$I_1'(0) = \int_0^l \mu\left\{2k_g'' - 2k_n\tau_g' - 4k_n'\tau + k_g\left(-k_g^2(l) + k_n^2(l) - k_g^2 + k_n^2 + 2\tau_g^2\right)\right\} ds + 2\mu'(l)k_g(l) - 2\mu(l)k_g'(l) + 4\mu(l)k_n(l)\tau_g(l) \tag{29}$$

and from (25)

$$I_2'(0) = \int_0^l \mu\left[p\left(\frac{\partial w}{\partial u}\right) + q\left(\frac{\partial w}{\partial v}\right) - k_g\left(w - w(l)\right)\right] ds. \tag{30}$$

From Eqs. (18), (29), (30), for all choices of the function $\mu(s)$ satisfying (7), with arbitrary values of $\mu(l)$ and $\mu'(l)$, and $K'(0) = 0$, two boundary conditions and differential equation that $\alpha(s)$ must satisfy as the minimum-energy trajectory is given as following:

(1) $k_g(l) = 0$,

(2) $k_g'(l) = 2k_n(l)\tau_g(l)$,

(3) $$\frac{1}{2}b\left[2k_g'' - 2k_n\tau_g' - 4k_n'\tau_g + k_g\left(-k_g^2(l) + k_n^2(l) - k_g^2 + k_n^2 + 2\tau_g^2\right)\right] - \gamma\left[p\left(\frac{\partial w}{\partial u}\right) + q\left(\frac{\partial w}{\partial v}\right) - k_g\left(w - w(l)\right)\right] = 0. \tag{31}$$

(ii) In the case of $k_g^2 > k_n^2$,

$$I_1'(0) = \int_0^l \mu\left\{-2k_g'' + 2k_n\tau_g' + 4k_n'\tau_g + k_g\left(k_g^2(l) - k_n^2(l) + k_g^2 - k_n^2 - 2\tau_g^2\right)\right\} ds - 2\mu'(l)k_g(l) + 2\mu(l)k_g'(l) - 4\mu(l)k_n(l)\tau_g(l). \tag{32}$$

From Eqs. (18), (32), (30), for all choices of the function $\mu(s)$ satisfying (7), with arbitrary values of $\mu(l)$ and $\mu'(l)$, and $K'(0) = 0$, the given arc α must satisfy two boundary conditions and differential equation

(1) $k_g(l) = 0$,

(2) $k_g'(l) = 2k_n(l)\tau_g(l)$,

(3) $$\frac{1}{2}b\left[-2k_g'' + 2k_n\tau_g' + 4k_n'\tau_g + k_g\left(k_g^2(l) - k_n^2(l) + k_g^2 - k_n^2 - 2\tau_g^2\right)\right] - \gamma\left[p\left(\frac{\partial w}{\partial u}\right) + q\left(\frac{\partial w}{\partial v}\right) - k_g\left(w - w(l)\right)\right] = 0. \tag{33}$$

2.2.1. Intrinsic equations for a elastic line deformed on an spacelike surface

(i) In the case of $k_g^2 < k_n^2$

$$I_1'(0) = \int_0^l \mu\{-2k_g'' - 2k_n\tau_g' - 4k_n'\tau_g + k_g(-k_g^2(l) + k_n^2(l) - k_g^2 + k_n^2 - 2\tau_g^2)\}\,ds \\ - 2\mu'(l)k_g(l) + 2\mu(l)k_g'(l) + 4\mu(l)k_n(l)\tau_g(l) \tag{34}$$

and from Eq. (25)

$$I_2'(0) = \int_0^l \mu\left[p\left(\frac{\partial w}{\partial u}\right) + q\left(\frac{\partial w}{\partial v}\right) - k_g(w - w(l))\right]ds. \tag{35}$$

From Eqs. (18), (34), (35), for all choices of the function $\mu(s)$ satisfying (7), with arbitrary values of $\mu(l)$ and $\mu'(l)$, and $K'(0) = 0$, the given arc α must satisfy two boundary conditions and differential equation

$$\begin{aligned} &(1)\quad k_g(l) = 0,\\ &(2)\quad k_g'(l) = -2k_n(l)\tau_g(l),\\ &(3)\quad \frac{1}{2}b[-2k_g'' - 2k_n\tau_g' - 4k_n'\tau_g + k_g(-k_g^2(l) + k_n^2(l) - k_g^2 + k_n^2 - 2\tau_g^2)]\\ &\qquad - \gamma\left[p\left(\frac{\partial w}{\partial u}\right) + q\left(\frac{\partial w}{\partial v}\right) - k_g(w - w(l))\right] = 0. \end{aligned} \tag{36}$$

(ii) In the case of $k_g^2 > k_n^2$

$$I_1'(0) = \int_0^l \mu\{2k_g'' + 2k_n\tau_g' + 4k_n'\tau_g + k_g(k_g^2(l) - k_n^2(l) + k_g^2 - k_n^2 + 2\tau_g^2)\}\,ds \\ + 2\mu'(l)k_g(l) - 2\mu(l)k_g'(l) - 4\mu(l)k_n(l)\tau_g(l). \tag{37}$$

From Eqs. (18), (37), (35), for all choices of the function $\mu(s)$ satisfying (7), with arbitrary values of $\mu(l)$ and $\mu'(l)$, and $K'(0) = 0$, the given arc α must satisfy two boundary conditions and differential equation

$$\begin{aligned} &(1)\quad k_g(l) = 0,\\ &(2)\quad k_g'(l) = -2k_n(l)\tau_g(l),\\ &(3)\quad \frac{1}{2}b[2k_g'' + 2k_n\tau_g' + 4k_n'\tau_g + k_g(k_g^2(l) - k_n^2(l) + k_g^2 - k_n^2 + 2\tau_g^2)]\\ &\qquad - \gamma\left[p\left(\frac{\partial w}{\partial u}\right) + q\left(\frac{\partial w}{\partial v}\right) - k_g(w - w(l))\right] = 0. \end{aligned} \tag{38}$$

We can give an example to a timelike plane. If Q is timelike, N is spacelike,

$$E = -1, \qquad G = 1.$$

$\mathfrak{L}$ and $\mathfrak{N}$ are components of the second fundamental form of the semi-Euclidean surface. From Definition 1.4,

$$\mathfrak{L} = \varepsilon_3\langle x_{uu}, N\rangle = 0, \qquad \mathfrak{N} = \varepsilon_3\langle x_{vv}, N\rangle = 0,$$

and

$$u' = \frac{\cosh\gamma}{|E|^{1/2}} = \cosh\gamma, \qquad v' = \frac{\sinh\gamma}{|G|^{1/2}} = \sinh\gamma.$$

Thus, from (5)

$$p = \sinh\gamma, \qquad q = -\cosh\gamma$$

and from Definition 1.4

$$k_g = -\gamma', \qquad \tau_g = 0.$$

Thus in timelike plane when α is timelike, (28) is written as following:

$$2\gamma''' - \gamma'^3 - \big[a(w - w_l)\big]\gamma' + a\left[\left(\frac{\partial w}{\partial u}\right)\sinh\gamma + \left(\frac{\partial w}{\partial v}\right)\cosh\gamma\right] = 0,$$

here $a = \frac{2\gamma}{b}$ denotes of the field strength to the bending resistance in Minkowski 3-space.

From the boundary conditions in (28),

$$\gamma'(l) = 0, \qquad \gamma''(l) = 0.$$

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